Suggested Solution of Revision Exercise 1

Question 1. Suppose that $\lim a_n = 3$. Using the definition of limit, show that

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10.$$

Solution. Note that for any $n \in \mathbb{N}$,

$$\left|\frac{a_n^2+1}{a_n-2} - 10\right| = \left|\frac{a_n^2-10a_n+21}{a_n-2}\right| = \frac{|a_n-7|}{|a_n-2|}|a_n-3|.$$

Since $\lim a_n = 3$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - 3| < \frac{1}{2}, \quad \forall n \geq N_1.$$

i.e., $2.5 < a_n < 3.5$. Hence for any $n \geq N_1$, we have

$$0.5 < |a_n-2| < 1.5 \quad \text{and} \quad 3.5 < |a_n - 7| < 4.5.$$

This implies that

$$\frac{|a_n-7|}{|a_n-2|} < \frac{4.5}{0.5} = 9, \quad \forall n \geq N_1.$$

Let $\varepsilon > 0$. Since $\lim a_n = 3$, there exists $N_2 \in \mathbb{N}$ such that

$$|a_n - 3| < \frac{\varepsilon}{9}, \quad \forall n \geq N_2.$$

Take $N = \max\{N_1, N_2\}$. Then for any $n \geq N$,

$$\left|\frac{a_n^2+1}{a_n-2} - 10\right| = \frac{|a_n-7|}{|a_n-2|}|a_n-3| < 9 \cdot \frac{\varepsilon}{9} = \varepsilon.$$

The result follows.

Question 2. Let (x_n) be a sequence of non-negative numbers. Suppose that

$$\lim_{n\to\infty} (-1)^n x_n$$

exists in $\mathbb{R}$. Show that (x_n) converges and find its limit.

Solution. Since $((-1)^n x_n)$ is convergent, its subsequences $((-1)^{2n} x_{2n})$ and $((-1)^{2n-1} x_{2n-1})$ are both convergent and have the same limit. Note that for each n,

$$(-1)^{2n} x_{2n} = x_{2n} \geq 0 \quad \text{and} \quad (-1)^{2n-1} x_{2n-1} = -x_{2n-1} \leq 0.$$

It follows that

$$0 \leq \lim((-1)^{2n} x_{2n}) = \lim((-1)^n x_n) = \lim(-1)^{2n-1} x_{2n-1} \leq 0.$$

Hence $\lim_{n\to\infty} (-1)^n x_n = 0$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |(-1)^n x_n| = \left|\lim_{n\to\infty} (-1)^n x_n\right| = 0.$$

Question 3. Let (x_n) be a bounded sequence of real numbers. Define

$$E = \{x \in \mathbb{R} : \text{there is a subsequence } (x_{n_k}) \text{ of } (x_n) \text{ that converges to } x.\}$$

Let $\alpha = \overline{\lim}\, x_n$. Show that $\alpha \in E$ and $\alpha = \sup E$.

Solution. Recall the definition of **limit superior**. i.e.,

$$\alpha = \overline{\lim}\, x_n = \lim_k \left(\sup_{n \geq k} x_n \right) = \lim_k u_k, \quad \text{where } u_k = \sup_{n \geq k} x_n.$$

To show that $\alpha \in E$, we need to find a subsequence (x_{n_k}) such that it converges to α. In other words, we need to pick an strictly increasing sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$. This will be done by choosing n_k's inductively. Write $n_0 = 1$. For each $k \in \mathbb{N}$, consider $u_{n_{k-1}+1} = \sup_{n > n_{k-1}} x_n$. By definition of supremum, there exists $n_k > n_{k-1}$ such that

$$u_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq u_{n_{k-1}+1}. \tag{1}$$

Notice that $\alpha = \lim_n u_n$ and $(u_{n_{k-1}+1})$ is a subsequence of (u_n). Hence $(u_{n_{k-1}+1})$ converges to α. Therefore the left expression and right expression in (1) both converge to α. By **Squeeze theorem**, it follows that

$$\alpha = \lim x_{n_k}.$$

To show that $\alpha = \sup E$, first note that $\alpha \leq \sup E$ because we have shown that $\alpha \in E$. On the other hand, let $x \in E$ and (x_{n_k}) be a subsequence of (x_n) that converges to x. Since $n_k \geq k$ for all $k \in \mathbb{N}$,

$$x_{n_k} \leq u_k, \quad \forall k \in \mathbb{N}.$$

Hence $x \leq \alpha$ after taking limits. Since $x \in E$ is arbitrary, $\sup E \leq \alpha$.

Question 4. Let a be a positive real number and $x_1 > \sqrt{a}$. Define the sequence (x_n) by

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

Prove that (x_n) is convergent and $\lim x_n = \sqrt{a}$.

Solution. We first show that $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$ by induction. The case for $n = 1$ is given. Note that

$$x_{n+1}^2 = \frac{1}{4} \left(x_n + \frac{a}{x_n} \right)^2 = \frac{1}{4} \left(x_n - \frac{a}{x_n} \right)^2 + a \geq a.$$

Since obviously $x_n \geq 0$ for all $n \in \mathbb{N}$, it follows that $x_{n+1} \geq \sqrt{a}$. Now, for any $n \in \mathbb{N}$,

$$x_{n+1} - x_n = \frac{1}{2} \left(\frac{a}{x_n} - x_n \right) \leq \frac{1}{2} \left(\frac{a}{\sqrt{a}} - \sqrt{a} \right) = 0.$$

Hence (x_n) is a decreasing sequence and it is bounded below by $\sqrt{a}$. By Monotone Convergence Theorem, (x_n) is convergent. Let $x = \lim x_n$. Then

$$x = \frac{1}{2} \left(x + \frac{a}{x} \right).$$

Solving gives $x = \pm\sqrt{a}$. Since $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$, $x \geq \sqrt{a}$. It follows that $x = \sqrt{a}$.

Question 5. .

(a) State the **Bolzano-Weierstrass Theorem**.

(b) State the the **Nested Intervals Property**.

(c) Using the **Bolzano-Weierstrass Theorem**, prove the **Nested Intervals Property**.

Solution. .

(a) A bounded sequence of real numbers has a convergent subsequence.

(b) Suppose $I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals. i.e.,

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots .$$

Then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$. i.e., $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Furthermore, if $\lim(b_n - a_n) = 0$, then the number ξ is unique.

(c) For the first part of the theorem, we need to find $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$ by Bolzano-Weierstrass Theorem. Consider the sequence (a_n) of real numbers. By Bolzano-Weierstrass Theorem, there exists a subsequence (a_{n_k}) of (a_n) that converges to some $\xi \in \mathbb{R}$. We need to show that $\xi \in I_m$ for all $m \in \mathbb{N}$.

Let $m \in \mathbb{N}$. There exists $K \in \mathbb{N}$ such that $n_K \geq m$. Hence

$$a_m \leq a_{n_K} \leq a_{n_k} \leq b_{n_k} \leq b_{n_K} \leq b_m, \quad \forall k \geq K.$$

In particular, we have

$$a_m \leq a_{n_k} \leq b_m, \quad \forall k \geq K.$$

Taking limits at $k \to \infty$, it follows that $a_m \leq \xi \leq b_m$. i.e., $\xi \in I_m$.

For the second part of the theorem, we further assume that $\lim(b_n - a_n) = 0$. We need to show that whenever $\eta \in I_n$ for all $n \in \mathbb{N}$, then $\eta = \xi$. This can be seen by observing

$$a_{n_k} \leq \eta \leq b_{n_k} \implies 0 \leq \eta - a_{n_k} \leq b_{n_k} - a_{n_k}, \quad \forall k \in \mathbb{N}$$

and then take limits.